ON THE STABILITY OF FIRST-ORDER NONLINEAR EQUATIONS OF NEUTRAL TYPE

PMM Vol. 34, №4, 1970, pp. 587 - 594
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(Received March 27, 1969)

The sufficient conditions of stability of the trivial solutions of first-order nonlinear equations of neutral type with arbitrary (finite or infinite) lag are established.

1. Let us consider the conditions of stability of the trivial solutions of equations of the form

$$x^{\cdot}(t) = -\int_0^\infty x(t-s)\,dK_0(s) + \sum_{n=1}^\infty a_n x^{\cdot}(t-h_n) + \int_0^\infty x^{\cdot}(t-s)\lambda(s)\,ds + b(t, x(t+\tau)), \qquad t > 0 \tag{1.1}$$

The stability of solutions of certain equations of the form (1.1) has been investigated in several studies. The conditions of stability of the solutions of these equations for $b(t, x(t+\tau)) \equiv 0$, $\lambda(s) \equiv 0$, $a_n \equiv 0$ were obtained in [1]. In [2, 3] the sufficient conditions of stability for lagging equations with $a_n \equiv 0$, $\lambda(s) \equiv 0$ were obtained by constructing Liapunov functionals.

The application of Liapunov's second method to equations of neutral type entails specific difficulties having to do with the dependence of the right sides of these equations on the derivatives of the solutions. The authors of [4] formulated the general theorems of Liapunov's second method for equations of the form

$$x^{\cdot}(t) = f(t, x(t-\tau(t)), \qquad x^{\cdot}(t-\tau(t))$$

In the present study we establish the conditions of stability of the trivial solutions of Eqs. (1.1) with the aid of Liapunov's method of functionals as developed in [5] for systems with lag. We note, however, that the functionals constructed below are not Liapunov functionals in the strict sense, being merely of constant rather than of fixed sign. The use of such sign-constant functionals makes it possible to circumvent the aforementioned difficulties presented by equations of neutral type and reduces the problem of establishing the stability of Eq. (1.1) to the solution of two ancillary problems.

The first problem consisiting in constructing a nonnegative functional with a negative-definite derivative along the trajectories of system (1.1).

The second problem consists in analyzing the stability of the solution $x(t) \equiv 0$ of the following inequality:

$$\left| x(t) - \sum_{n=1}^\infty a_n x(t-h_n) - \int_0^\infty x(t-s)\lambda(s)\,ds \right| \leqslant c_0 \tag{1.2}$$

The symbols c_i here and below denote certain positive constants.

We also note that some of the distinctive features of the above problem have to do with the fact that the deviations of the argument can be infinite.

2. In the discussion below we assume that the coefficients of Eq. (1.1) satisfy the following requirements.

The variation of the kernel $K_0(s)$ is bounded over the semiaxis $[0, \infty)$ and the corresponding integral in (1.1) must be construed in the Stieltjes sense. The function $\lambda(s)$ is bounded and Riemann-integrable over $[0, \infty]$.

Finally, all the constants

$$h_n \geqslant 0, \qquad |a_1| + |a_2| + \ldots + |a_n| + \ldots < \infty$$

Let us agree that

$$K_1(s) = \int_0^s \lambda(s_1)\, ds_1 + \sum_{h_n \leqslant s} a_n, \qquad K_1(0) = 0$$

where summation is carried out over those values of n for which $h_n \leqslant s$, and set

$$\sum_{n=1}^{\infty} a_n y(-h_n) + \int_0^{\infty} y(-s)\lambda(s)\, ds = \int_0^{\infty} y(-s)\, dK_1(s) \tag{2.1}$$

for any bounded Riemann-integrable bounded function $y(s)$, $s \leqslant 0$. We note that the symbol in the right side of (2.1) is generally not a Stiltjes integral, since this integral may not exist under the above assumptions concerning $y(s)$.

We denote by Q the direct product of the semiaxis $[0, \infty)$ and the space $C(-\infty, 0]$ of continuous functions $\varphi(\tau)$ of the argument τ bounded on the semiaxis $(-\infty, 0]$; the argument τ varies in the range $-\infty < \tau \leqslant 0$.

The metric in Q is defined by the formula

$$\rho((t_1, \varphi_1), (t_2, \varphi_2)) = |t_1 - t_2| + \sup_{\tau \leqslant 0} |\varphi_1(\tau) - \varphi_2(\tau)|$$
$$\varphi_1, \varphi_2 \in C(-\infty, 0], \qquad t_1, t_2 \geqslant 0$$

The functional $b(t, \varphi(\tau))$ is defined and continuous on the space Q and satisfies the conditions

$$b(t, 0) \equiv 0, \; |b(t, \varphi) - b(t, \psi)|^2 \leqslant \int_0^{\infty} |\varphi(-s) - \psi(-s)|^2\, dK_2(s) \tag{2.2}$$

(where the function $K_2(s)$ is monotonically nondecreasing) for all $\varphi, \psi \in C(-\infty, 0]$. Let us set

$$\alpha_{ij} = \int_0^{\infty} s^i |dK_j(s)| \qquad (i = 0, 1, \ldots, j = 0, 1, 2, 3, 4)$$

From now on we assume that

$$\alpha_{00} < \infty, \qquad \alpha_{01} < 1, \qquad \alpha_{02} < \infty \tag{2.3}$$

The solution $x(t)$ of Eq. (1.1) for $t > 0$ is determined by the initial conditions

$$x(t) = \varphi(t), \; x^{\cdot}(t) = \psi(t), \; t \leqslant 0 \tag{2.4}$$

From now on we shall limit ourselves to initial conditions which satisfy Conditions (A): $\varphi(t)$, $t \leqslant 0$ is a continuous bounded function; $\psi(t)$ is a function bounded on $(-\infty, 0]$ and is Riemann-integrable over every finite interval $\psi(t) = \varphi^{\cdot}(t)$ almost everywhere.

To emphasize the dependence of the solution $x(t)$ of problem (1.1), (2.4) on the initial conditions we shall occasionally denote it by the symbol $x(t, \varphi, \psi)$. Under assumptions (2.2)–(2.4) the method of successive approximations (such as that used to prove Theorem 2 of [6] affords a ready means of proving the existence and uniqueness of the function $x^{\cdot}(t)$ bounded and Riemann-integrable over every finite interval, equal to $\psi(t)$ for $t \leqslant 0$, and such that the functions

$$x^{\cdot}(t), \qquad x(t) = \varphi(0) + \int_0^t x^{\cdot}(s)\,ds$$

are the solution of problem (1.1), (2.4).

3. Definition 1. We call the trivial solution of Eq. (1.1) –

1) "stable" if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|x(\varphi, \psi, t)| < \varepsilon$ for all $t \geqslant 0$ provided that Conditions (A) are fulfilled and that

$$\|\varphi\| = \sup |\varphi(\tau)| < \delta(\varepsilon) \qquad (\tau \leqslant 0)$$

2) "asymptotically stable" if it is stable and

$$\lim_{t\to\infty} x(\varphi, \psi, t) = 0$$

Now let us prove a simple lemma concerning the stability of the trivial solution of inequality (1.2) which we shall need below.

Lemma 1. Let the function $x(t) = \varphi(t)$ for $t \leqslant 0$ ($\varphi(t) \in C(-\infty, 0]$); let it satisfy inequality (1.2) for $t > 0$, and let condition (2.3) be fulfilled. The estimate

$$|x(t)|(1 - \alpha_{01}) \leqslant c_0 + \alpha_{01}\|\varphi\|, \qquad t \geqslant 0$$

is then valid.

Proof. Let us introduce the function

$$\rho(t) = \max|x(s)|, \quad 0 \leqslant s \leqslant t$$

From (1.2) we obtain the relation

$$|x(t)| \leqslant c_0 + \alpha_{01}(\rho(t) + \|\varphi\|)$$

This implies that

$$\rho(t)(1 - \alpha_{01}) \leqslant c_0 + \alpha_{01}\|\varphi\|$$

Lemma 1 has been proved.

Let us set

$$Z(\varphi(\tau)) = \varphi(0) - \sum_{n=1}^{\infty} a_n \varphi(-h_n) - \int_0^{\infty} \varphi(-s)\lambda(s)\,ds = \varphi(0) - \int_0^{\infty} \varphi(-s)\,dK_1(s) \tag{3.1}$$

for any $\varphi(\tau) \in C(-\infty, 0]$.

Theorem 1. The trivial solution of Eq. (1.1) is asymptotically stable if conditions (2.2), (2.3) are fulfilled and if there exists a functional

$$V(\varphi(\tau)) = W(\varphi(\tau)) + Z^2(\varphi(\tau)) \tag{3.2}$$

defined on the space Q which satisfies Lipschitz's local condition (whereby for any N there exists an m_N such that $|V(\varphi(\tau)) - V(\psi(\tau))| \leqslant m_N\|\varphi - \psi\|$ for $\|\varphi(\tau)\| \leqslant N$, $\|\psi(\tau)\| \leqslant N$); moreover,

$$0 \leqslant W(\varphi(\tau)) \leqslant \omega_1(\|\varphi\|) \tag{3.3}$$

and the derivative of functional (3.2) computed along the trajectories of system (1.1) exists and satisfies the condition

$$\frac{dV(x(t+\tau))}{dt} \leqslant -\omega_2(|x(t)|) \tag{3.4}$$

for almost all $t \geqslant 0$ (in the Lebesgue measure). The functions $\omega_i(r)$ in estimates (3.3) and (3.4) are continuous, and $\omega_i(0) = 0$, $\omega_i(r) > 0$ for $r > 0$.

Proof. We infer from (3.1) – (3.3) that

$$V(\varphi(\tau)) \leqslant \omega_1(\|\varphi\|) + (1+\alpha_{01})^2 \|\varphi\|^2 \tag{3.5}$$

Further, Lipschitz's local condition and the aforementioned boundedness of the derivative $x^{\cdot}(t)$ of the solution $x(t)$ over every bounded interval imply that the functional $V(x(t+\tau))$ is continuous as a function of t. This means that for all nonnegative t_1, t_2 we have

$$V(x(t_2+\tau)) - V(x(t_1+\tau)) = \int_{t_1}^{t_2} \frac{dV(x(t+\tau))}{dt}\, dt$$

Hence, by virtue of (3.2) – 3.4),

$$Z^2(x(t+\tau)) \leqslant V(x(t+\tau)) \leqslant V(\varphi(\tau))$$

This result, expression (3.5), and Lemma 1 imply that

$$|x(t)| \leqslant \frac{1}{1-\alpha_{01}} [(\omega_1(\|\varphi\|) + (1+\alpha_{01})^2\|\varphi\|^2)^{1/2} + \alpha_{01}\|\varphi\|]$$

The stability of the trivial solution of Eq. (1.1) has thus been established.

To prove asymptotic stability we need only note that by virtue of Lemma 1 the modulus $|x^{\cdot}(t)|$ for any bounded solution $x(t)$ of problem (1.1), (2.4) is also bounded, and then repeat the argument of Krasovskii ([5], p. 181).

Theorem 1 has been proved.

4. Theorem 2. Let conditions (2.3) be fulfilled and let the kernel $K_0(s)$ have a discontinuity of magnitude $a > 0$ at zero; moreover, let

$$a(1-\alpha_{01}) > (1+\alpha_{01})\Big(\int_{+0}^{\infty} |dK_0(s)| + \sqrt{\alpha_{02}}\Big) \tag{4.1}$$

$$\alpha_{12} + \alpha_{11} + \alpha_{10} < \infty$$

The trivial solution of Eq. (1.1) is then asymptotically stable.

Proof. We need merely show that the requirements of Theorem 2 enable us to construct a functional which satisfies the conditions of Theorem 1. We begin by assuming that $\alpha_{02} = 0$ (which means that $b(t, \varphi(\tau)) \equiv 0$) and consider the functional

$$V_0(x(t+\tau)) = Z^2(x(t+\tau)) + \alpha_{00} \int_0^{\infty} |dK_1(s)| \int_{t-s}^{t} x^2(s_1)\, ds_1 + \\ + (1+\alpha_{01}) \int_{+0}^{\infty} |dK_0(s)| \int_{t-s}^{t} x^2(s_1)\, ds_1 \tag{4.2}$$

The validity of estimates (3.3) follows from the inequality

$$\int_0^{\infty} |dK_i(s)| \int_{t-s}^{t} x^2(s_1)\, ds_1 \leqslant \alpha_{1i} \|x(t+\tau)\|^2 \quad (i=0,1) \tag{4.3}$$

Let us compute the derivative of functional (4.2) along the trajectories of system (1.1). Since

$$Z^{\cdot}(x(t+\tau)) = x^{\cdot}(t) - \int_0^{\infty} x^{\cdot}(t-s)\, dK_1(s) =$$

$$= -a\,x(t) - \int_{+0}^{\infty} x(t-s)\, dK_0(s) + b(t, x(t+\tau))$$

for almost all $t \geqslant 0$, it follows that

$$\frac{dV_0(x(t+\tau))}{dt} = -2Z(x(t+\tau))\Big[a\,x(t) + \int\limits_{+0}^{\infty} x(t-s)\,dK_0(s) + b(t, x(t+\tau))\Big] - \alpha_{00}\int\limits_0^{\infty} x^2(t-s)\,|dK_1(s)| - (1+\alpha_{01})\int\limits_{+0}^{\infty} x^2(t-s)\,|dK_0(s)| + x^2(t)\Big[\alpha_{00}\cdot\alpha_{01} + (1+\alpha_{01})\int\limits_{+0}^{\infty}|dK_0(s)|\Big]$$

Hence, making use of (2.2), (2.3) and the Cauchy-Buniakowski inequality, we find that

$$\frac{dV_0(x(t+\tau))}{dt} \leqslant 2x^2(t)\Big[-a + a\alpha_{01} + \int\limits_{+0}^{\infty}|dK_0(s)|\,(1+\alpha_{01})\Big]$$

for almost all $t > 0$.

Hence, functional (4.2) satisfies the requirements of Theorem 1. This proves the validity of Theorem 2 for $\alpha_{02} = 0$. If $\alpha_{02} > 0$ we need merely consider the functional

$$V(x(t+\tau)) = V_0(x(t+\tau)) + \frac{1}{\sqrt{\alpha_{02}}}\int\limits_0^{\infty}|dK_1(s)|\int\limits_{t-s}^{t} x^2(s_1)\,ds_1 + \left(\alpha_{01}\sqrt{\alpha_{02}} + \frac{1}{\sqrt{\alpha_{02}}}\right)\int\limits_0^{\infty} dK_2(s)\int\limits_{t-s}^{t} x^2(t_1)\,dt_1 \qquad (4.4)$$

As above, it is easy to show that functional (4.4) satisfies inequalities (3.3), and that its derivative satisfies the estimate

$$\frac{dV(x(t+\tau))}{dt} \leqslant 2x^2(t)\Big[-a(1-\alpha_{01}) + (1+\alpha_{01})\Big(\int\limits_{+0}^{\infty}|dK_0(s)| + \sqrt{\alpha_{02}}\Big)\Big]$$

Theorem 2 has been proved.

5. Lemma 2. Let requirements (2.2), (2.3) and the condition $\alpha_{01} + \alpha_{10} < 1$ be fulfilled. The statement of Theorem 1 then remains valid if there exists a functional of the form

$$V[\varphi(\tau)] = Z_1^2(\varphi(\tau)) + W[\varphi(\tau)] \qquad (5.1)$$

where

$$Z_1(\varphi(\tau)) = \varphi(0) - \int\limits_0^{\infty}\varphi(-s)\,dK_1(s) - \int\limits_0^{\infty} dK_0(s)\int\limits_{-s}^{0}\varphi(t_1)\,dt_1 \qquad (5.2)$$

which satisfies (3.3), (3.4).

This lemma can be proved by repeating verbatim the proof of Theorem 1.

Theorem 3. Let conditions (2.2), (2.3) be fulfilled and let

$$\alpha_{10} + \alpha_{01} < 1$$

$$\beta = \int\limits_0^{\infty} dK_0(s) > \alpha_{02}^{1/2}\,\frac{1+\alpha_{10}+\alpha_{01}}{1-\alpha_{10}-\alpha_{01}}$$

$$\alpha_{20} + \alpha_{11} + \alpha_{12} < \infty$$

The trivial solution of Eq. (1.1) is then asymptotically stable.

Proof. We begin by assuming that $\alpha_{02} > 0$ and introducing the functional

$$V[x(t+\tau)] = Z_1^2(x(t+\tau)) + (\beta + \alpha_{02}^{1/2}) \int_0^\infty |dK_1(s)| \int_{t-s}^t x^2(t_1)\,dt_1 + \tag{5.3}$$

$$+ (1+\alpha_{10}+\alpha_{01})\alpha_{02}^{-1/2} \int_0^\infty dK_2(s) \int_{t-s}^t x^2(t_1)\,dt_1 + (\beta + \alpha_{02}^{1/2}) \int_0^\infty |dK_0(s)| \int_{t-s}^t dt_1 \int_{t_1}^t x^2(t_2)\,dt_2$$

where $Z_1(x(t+\tau))$ is defined by formula (5.2). Inequality (4.3) and the conditions of Theorem 3 imply that functional (5.3) satisfies the estimates

$$Z_1^2(x(t+\tau)) \leqslant V[x(t+\tau)] \leqslant C_1 \| x(t+\tau) \|^2 \tag{5.4}$$

The derivative

$$Z_1^{\cdot}(x(t+\tau)) = -\beta x(t) + b(t, x(t+\tau)).$$

for almost all $t > 0$. Moreover, expressions (2.2), (2.3) yield the estimates

$$2b(t, x(t+\tau)) \int_0^\infty dK_0(s) \int_{t-s}^t x(t_1)\,dt_1 \leqslant$$

$$\leqslant \alpha_{10}\alpha_{01}^{-1/2} \int_0^\infty x^2(t-s)\,dK_2(s) + \alpha_{02}^{1/2} \int_0^\infty |dK_0(s)| \int_{t-s}^t x^2(t_1)\,dt_1$$

$$2\beta x(t) \int_0^\infty dK_0(s) \int_{t-s}^t x(t_1)\,dt_1 \leqslant \beta x^2(t)\alpha_{00} + \beta \int_0^\infty |dK_0(s)| \int_{t-s}^t x^2(t_1)\,dt_1$$

From this we readily infer that

$$\frac{dV[x(t+\tau)]}{dt} \leqslant 2x^2(t)[-\beta(1-\alpha_{01}-\alpha_{10}) + \alpha_{02}^{1/2}(1+\alpha_{10}+\alpha_{01})]$$

for almost all $t > 0$.

This fact, expression (5.4), and Lemma 2 for $\alpha_{02} > 0$ imply the validity of Theorem 3 in this case.

To prove Theorem 3 for $\alpha_{02} = 0$ we must make use of functional (5.3) minus its third term; α_{02} must be set equal to zero in the remaining terms.

Note 1. If $\lambda(s) \equiv 0$, $a_n \equiv 0$, Theorems 2 and 3 become the corresponding results of [3].

Note 2. By modifying functional (5.3), we can obtain other conditions of stability of Eq. (1.1). Let us cite some of these conditions. We can make use of the fact that an arbitrary function $K_0(s)$ with bounded variation can be expressed as the difference $K_0(s) = K_3(s) - K_4(s)$ between two monotonically nondecreasing bounded functions. Here we need merely add the expression

$$(1+\alpha_{01}+\alpha_{13}) \int_0^\infty dK_4(s) \int_{t-s}^t x^2(t_1)\,dt_1 +$$

$$+ \alpha_{04} \int_0^\infty |dK_1(s)| \int_{t-1}^t x^2(t_1)\,dt_1 + \alpha_{04} \int_0^\infty dK_3(s) \int_{t-s}^t dt_1 \int_{t_1}^t x(t_2)\,dt_2$$

to functional (5.3) in which $K_0(s)$ has been replaced by $K_3(s)$, β by α_{03} and α_{10} by α_{13}.

This new functional enables us to prove

Theorem 4. Let conditions (2.2), (2.3) be fulfilled and let

$$\alpha_{01} + \alpha_{13} < 1$$

$$\alpha_{03} > [\alpha_{04} + \alpha_{02}^{1/2}] \frac{1 + \alpha_{01} + \alpha_{13}}{1 - \alpha_{01} - \alpha_{13}}$$

$$\alpha_{11} + \alpha_{12} + \alpha_{14} + \alpha_{23} < \infty$$

The trivial solution of Eq. (1.1) is then asymptotically stable.

Example. Let us consider the equation

$$x^{\cdot}(t) = -r_1 \int_0^{\infty} x(t-s) e^{-s} ds + r_2 x^{\cdot}(t-h) \qquad (t>0) \tag{5.5}$$

where r_1, r_2, $h \geqslant 0$ are some constants. According to Theorem 3 the trivial solution of Eq. (5.5) is asymptotically stable if

$$r_1 > 0, \ |r_2| + r_1 < 1$$

6. By altering slightly functionals (4.2), (5.3) which we constructed in proving Theorems 2 – 4, we can obtain the conditions of stability of the trivial solutions of the equations

$$x^{\cdot}(t) = -\int_0^{\infty} x(t-s)\, d_s R_0(t, s) + \sum_{n=1}^{\infty} a_n(t)\, x^{\cdot}(t - h_n(t)) +$$

$$+ \int_0^{\infty} x^{\cdot}(t-s)\, \lambda(t, s)\, ds + b(t, x(t+\tau)) \tag{6.1}$$

Since this entails some very cumbersome functionals, we shall consider only the simplest cases; these, however, suffice for indicating the alterations which must be made in functionals (4.2), (5.3) in considering more general equations of the form (6.1). Let us find the conditions of stability of the trivial solution of the equation

$$x^{\cdot}(t) = -b(t)\, x(t-h) + cx^{\cdot}(t-h), \ \ t > 0 \tag{6.2}$$

The solution of Eq. (6.2) for $t > 0$ is determined by initial conditions (2.4), and stability must be construed in the sense of Definition 1. We assume that the function $b(t)$ is continuous and nonnegative. The functional

$$V[x(t-s)] = \Big[x(t) - cx(t-h) - \int_{t-h}^{t} b(s+h)\, x(s)\, ds\Big]^2 +$$

$$+ |c| \int_{t-h}^{t} b(s+2h)\, x^2(s)\, ds + \int_{t-h}^{t} b(t_1 + 2h)\, dt_1 \int_{t_1}^{t} b(t_2 + h)\, x^2(t_2)\, dt_2$$

enables us to show (as we did in proving Theorem 3) that the trivial solution of Eq. (6.2) is asymptotically stable if

$$\sup_{t\geqslant 0} \Big\{|c| + \int_{t-h}^{t} b(s+h)\, ds\Big\} < 1$$

$$\sup_{t\geqslant 0} \Big\{-2b(t+h) + |c|\,[b(t+h) + b(t+2h)] +$$

$$+ b(t+h) \int_{t-h}^{t} [b(s+h) + b(s+2h)]\, ds\Big\} < 0$$

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Translated by A. Y.

THE MOTION OF A NONSYMMETRIC SELF-EXCITING GYROSTAT

PMM Vol. 34, №4, 1970, pp. 595-602
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The motion of a self-exciting gyrostat is investigated in the special case where the force moment of the housing acts along one of the axes of inertia of the gyrostat while the projection of the gyrostatic moment (the moment of the relative momentum of the internal flywheels) on this axis is equal to zero. The parameters of the problem are the force moment and the moments of momenta of the flywheels, which are all assumed to be constant. The dependence of the hodograph of the angular velocity vector of the gyrostat on these parameters is investigated; the domains of parameter values corresponding to various types of motion are determined.

Grammel [1, 2] investigated a similar problem for a solid without internal rotations. The present study constitutes an extension of this familiar case.

1. The initial relations. The motion of a gyrostat with a constant gyrostatic moment $\mathbf{h}$ under the action of an external moment $\mathbf{m}$ is described by the following system of equations:

$$\begin{aligned} A_1\omega_1^{\cdot} + (A_3 - A_2)\,\omega_2\omega_3 + \omega_2 h_3 - \omega_3 h_2 &= m_1 \\ A_2\omega_2^{\cdot} - (A_3 - A_1)\,\omega_3\omega_1 + \omega_3 h_1 - \omega_1 h_3 &= m_2 \\ A_3\omega_3^{\cdot} + (A_2 - A_1)\,\omega_1\omega_2 + \omega_1 h_2 - \omega_2 h_1 &= m_3 \end{aligned} \qquad (1.1)$$

Here A_1, A_2, A_3 are the moments of inertia of the gyrostat with respect to its principal central axes x_1, x_2, x_3. For definiteness we assume that $A_1 < A_2 < A_3$; ω_1, ω_2, ω_3 are the projections of the angular velocity vector of the gyrostat on the axes x_1, x_2, x_3; h_1, h_2, h_3 are the projections of the gyrostatic moment, and m_1, m_2, m_3 are